On the Number of Minimal Models of log Smooth 3folds (Cascini–Lazić)

Conjecture $/\mathbb{C}$

The number of minimal models of a smooth projective variety is finite up to isomorphism.

- Known for general type (BCHM)
- Known for 3folds with $K>0$ (Kawamata)

Today

look at a class of 3fold log smooth pairs (X,Δ)

$\rightsquigarrow$ bound the number of log terminal models of (X,Δ) by a constant dependent only on the <u>topological type</u> of (X,Δ).

Definitions & Tools

"geometric valuation" $\longleftrightarrow$ "divisorial valuation"

"log terminal model" $\longleftrightarrow$ "minimal model"

For (X,Δ) dlt a birational contraction

$$f: X \dashrightarrow Y$$

is a <u>log terminal model</u> if f is $(K_X+\Delta)$-negative, Y $\mathbb{Q}$-factorial, and $K_Y+f_*\Delta$ nef.

If $K_Y+f_*\Delta$ is semiample, we get a fibration $Y \to Z$, and the composite $X \dashrightarrow Z$ is called the <u>ample model</u> (canonical model)

$$\mathrm{MMP}(n) := \Big((X,\Delta) \text{ l.c.} + K_X + \Delta \text{ pseudoeffective} \Big)$$

$\Longrightarrow (X,\Delta)$ admits a log terminal and an ample model

X normal projective variety

For $D \in \mathrm{Div}_{\mathbb{R}}(X)$,

the stable base locus $\quad \mathbf{B}(D) := \bigcap_{D \sim_{\mathbb{R}} D' \geq 0} \mathrm{Supp}(D')$

$$\begin{array}{c}\mathrm{MMP}(n)\\ (X,\Delta) \text{ klt}\end{array} \Longrightarrow \left\{ \begin{array}{c} \text{prime divisors} \\ \text{contracted by } f : X \dashrightarrow Y \\ \text{log terminal model} \\ \text{of } (X,\Delta) \end{array} \right\} = \left\{ \begin{array}{c} \text{prime divisors} \\ \text{in } \mathbf{B}(K_X+\Delta) \end{array} \right\}$$

A, B pseudoeffective $\Longrightarrow \mathbf{B}(A+B) \subseteq \mathbf{B}(A) + \mathbf{B}(B)$

the augmented stable base locus $\quad \mathbf{B}_+(D) := \bigcap_{\varepsilon > 0} \mathbf{B}(D - \varepsilon A) \supseteq \mathbf{B}(D)$

for A some ample divisor on X.

Lemma 2.5. *Let X be a smooth projective variety and let D be a big $\mathbb{Q}$-divisor on X. Let $f : X \dashrightarrow Y$ be the ample model of D.*

Then $\mathbf{B}_-(D)$ coincides with the exceptional locus of f.

Main tools for computing discrepancies

Lemma 2.1. *Let $(X, \sum_{i=1}^{p} b_i S_i)$ be a log smooth terminal threefold pair, where $S_1, \ldots, S_p$ are distinct prime divisors. Let*

$$f : X \dashrightarrow X'$$

be a birational contraction to a terminal threefold X'. Let S_i' be the proper transform of S_i in X' for every i. Let Y be a smooth variety, let $g : Y \longrightarrow X$ be a birational morphism, and let $E \subseteq Y$ be an $(f \circ g)$-exceptional prime divisor such that the centre of E on X' is a curve. Then

$$a\left(E, X', \sum_{i=1}^{p} b_i S_i'\right) = a(E, X', 0) - \sum_{i=1}^{p} b_i \operatorname{mult}_E S_i', \tag{1}$$

where $a(E, X', 0)$ is an integer such that $0 < a(E, X', 0) \leqslant \rho(Y/X')$.

Lemma 2.2. *Let (X, Δ) be a canonical projective pair, and let $f : X \dashrightarrow Y$ be a $(K_X + \Delta)$-nonpositive birational contraction. Assume that f does not contract any component of Δ, and let $\Delta_Y = f_* \Delta$.*

Then (Y, Δ_Y) is canonical. Additionally, if f is $(K_X + \Delta)$-negative and (X, Δ) is terminal, then (Y, Δ_Y) is terminal.

Shokurov's log Geography

Definition 2.12. Let $(X, \sum_{i=1}^p S_i)$ be a log smooth projective pair, where $S_1, \dots, S_p$ are distinct prime divisors, and let $V = \sum_{i=1}^p \mathbb{R}S_i \subseteq \mathrm{Div}_{\mathbb{R}}(X)$. Let $0 < \varepsilon < 1/2$. We denote

$$\mathcal{L}(V) = \left\{ \sum_{i=1}^p a_i S_i \in V \;\middle|\; a_i \in [0,1] \right\}, \qquad \mathcal{E}(V) = \{\Delta \in \mathcal{L}(V) \mid K_X + \Delta \sim_{\mathbb{R}} D \geq 0\},$$

and

$$\mathcal{L}_\varepsilon(V) = \left\{ \sum_{i=1}^p a_i S_i \in V \;\middle|\; a_i \in [\varepsilon, 1-\varepsilon] \right\}, \qquad \mathcal{L}_\varepsilon^{can}(V) = \{\Delta \in \mathcal{L}_\varepsilon(V) \mid (X, \Delta) \text{ is canonical}\}.$$

$\mathcal{L}(V), \mathcal{L}_\varepsilon(V), \mathcal{L}_\varepsilon^{can}(V)$ rational polytopes

If $\dim \mathcal{L}_\varepsilon^{can}(V) = p$ and $\Delta \in \mathrm{int}(\mathcal{L}_\varepsilon^{can}(V))$, Δ is terminal

For $f: X \dashrightarrow Y$ a contraction,

$$\mathcal{C}_f(V) := \overline{\{\Delta \in \mathcal{E}(V) : f \text{ log terminal model of } (X, \Delta)\}}$$

Shokurov

Theorem 2.13. *Assume the MMP in dimension n. Let $(X, \sum_{i=1}^p S_i)$ be a log smooth projective pair, where $S_1, \dots, S_p$ are distinct prime divisors, and let $V = \sum_{i=1}^p \mathbb{R}S_i \subseteq \mathrm{Div}_{\mathbb{R}}(X)$.*

Then there exist birational contractions $f_i : X \dashrightarrow Y_i$ for $i = 1, \dots, k$, such that $\mathcal{C}_{f_1}(V), \dots, \mathcal{C}_{f_k}(V)$ are rational polytopes and

$$\mathcal{E}(V) = \bigcup_{i=1}^k \mathcal{C}_{f_i}(V).$$

In particular, $\mathcal{E}(V)$ is a rational polytope.

Theorem 2.14. *Assume the MMP in dimension n and the relative Cone conjecture in dimensions $\leqslant n$. Let X be a terminal projective variety of dimension n.*

Then the number of minimal models of X is finite up to isomorphism.

~~proof~~

$X \to$ replace with min. model

$X \to S$ canonical model

If Y is another minimal model

and $A \in \mathrm{Div}(Y)$ very ample $/S$

$\varphi : X \dashrightarrow Y$

is an isomorphism in codim 1.

$D := \varphi^* A$ is movable over S

and $Y \cong \mathrm{Proj}_S R(X/S, D)$

Π = fundamental domain for $\mathrm{Bir}(X/S) \curvearrowright \overline{\mathrm{Mov}}^e(X/S)$

$\Longrightarrow$ Have $g \in \mathrm{Bir}(X/S)$ s.t. $g^* D \in \Pi$

and $R(X/S, D) \cong R(X/S, g^*D)$
(g pseudo-automorphism)

Replace D with $g^*D \in \Pi$

Assume $D_1, \dots, D_r$ effective generating Π

$S_1, \dots, S_p$ all prime divisors in $\mathrm{Supp}\left(\sum_{i=1}^{r} D_i\right)$.

$$V = \sum_{i=1}^{p} \mathbb{R} \cdot S_i$$

$\Pi' =$ inverse image of Π in V

$$D \in \Pi' \cap \mathbb{R}_+ \mathcal{L}(V)$$

$K_X \sim_S 0 \xRightarrow{\text{Shokurov}}$ Have $\{(C_i, f_i) : f_i : X \dashrightarrow Z$ contraction$\}$ ($C_i \subseteq V$ cone)

s.t. $\Pi' \cap \mathbb{R}_+ \mathcal{L}(V) = \bigcup_{i=1}^{k} C_i$

and $\Delta \in C_i \cap \mathcal{L}(V) \iff f_i$ ample model of $K_X + \Delta$

$\Longrightarrow D \in C_i$ for some i, and $Y \cong Z_i$ $\square$

If $C_i := C_f(V) \cap \mathcal{L}_\varepsilon^{can}(V)$ is of dim p, we call it a terminal chamber.

Theorems we work toward

Theorem 1.1. *Let p and ρ be positive integers, and let ε be a positive rational number. Let $(X, \sum_{i=1}^{p} S_i)$ be a 3-dimensional log smooth pair such that*

(i) *X is not uniruled,*
(ii) *$S_1, \dots, S_p$ are distinct prime divisors which are not contained in $\mathbf{B}(K_X + \sum_{i=1}^{p} a_i S_i)$ for all $0 \leqslant a_i \leqslant 1$,*
(iii) *the divisors S_i span $\mathrm{Div}_{\mathbb{R}}(X)$ up to numerical equivalence.*
(iv) *$\rho(X) \leqslant \rho$ and $\rho(S_i) \leqslant \rho$ for all $i = 1, \dots, p$.*

Corollary 1.2. *Let ε be a positive number. Let $\mathfrak{X}$ be the collection of all log smooth 3-fold terminal pairs $(X, \Delta = \sum_{i=1}^{p} \delta_i S_i)$ such that X is not uniruled, $\varepsilon \leqslant \delta_i \leqslant 1 - \varepsilon$ for all i, $S_1, \dots, S_p$ are distinct prime divisors not contained in $\mathbf{B}(K_X + \sum_{i=1}^{p} a_i S_i)$ for all $0 \leqslant a_i \leqslant 1$, and S_i span $\mathrm{Div}_{\mathbb{R}}(X)$ up to numerical equivalence.*

Then for every $(X_0, \Delta_0) \in \mathfrak{X}$ there exists a constant N such that for every $(X, \Delta) \in \mathfrak{X}$ of the topological type as (X_0, Δ_0), the number of log terminal models of (X, Δ) is bounded by N.

Lemma 4.1. *Let $(X, S=\sum_{i=1}^{p} S_i)$ be a log smooth projective threefold, where $S_1,\dots,S_p$ are distinct prime divisors, and assume that $0<\varepsilon\leqslant 1/2$ is a rational number such that $(X,\varepsilon S)$ is terminal and $K_X+\varepsilon S$ is big. Assume that $S_i \not\subseteq \mathbf{B}_+(K_X+\varepsilon S)$ for every i. Let I be the total number of irreducible components of intersections of each two of the divisors $S_1,\dots,S_p$.*

Then for any i, the number of curves contained in

$$\mathbf{B}_+(K_X+\varepsilon S)\cap S_i$$

is bounded by a constant which depends on $\rho(X)$, $\rho(S_i)$, ε and I.

proof Fix $i \in \{1,\dots,p\}$

Have a sequence of $(K_X+\varepsilon S)$-flips and div. contractions

$$f: X = X^0 \dashrightarrow \cdots \dashrightarrow X^k \longrightarrow X^{k+1}$$

log terminal model of $(X,\varepsilon S)$

ample model of $(X,\varepsilon S)$

2.5 $\rightsquigarrow$ Since $K_X+\varepsilon S$ big, $\operatorname{exc}(f) = \mathbf{B}_+(K_X+\varepsilon S)$

For all $l \in \{1,\dots,p\}$ denote

S_l^j = proper transform of S_l in X^j

$\overline{S_l^j}$ = normalization

$S^j = \sum_{l=1}^{p} S_l^j$

$$g : S_i = S_i^0 \dashrightarrow \cdots \dashrightarrow S_i^k \dashrightarrow S_i^{k+1}$$

$$\bar{g} : S_i = \bar{S}_i^0 \dashrightarrow \cdots \dashrightarrow \bar{S}_i^k \dashrightarrow \bar{S}_i^{k+1}$$

Fix $C \subseteq B_+(K_X + \varepsilon S) \cap S_i \subseteq \operatorname{exc}(f)$

<u>Case 1</u>: g is an isomorphism at the gen. point of C

There exists $E \supseteq C$ a prime divisor of X

with $f(E) = f(C)$

Since $(X^{k+1}, \varepsilon S^{k+1})$ canonical and $f(C) \subseteq S^{k+1}$,

X^{k+1} is terminal at gen. point of $f(C)$

$$0 \le a(E, X^{k+1}, \varepsilon S^{k+1}) < \rho(X) - \varepsilon \operatorname{mult}_E S^{k+1}$$

$$\le \rho(X) - \varepsilon \operatorname{mult}_{f(E)} S^{k+1}$$

$$\Longrightarrow \operatorname{mult}_{f(E)} S_i^{k+1} < \rho(X)/\varepsilon$$

So, $\#\begin{pmatrix}\text{curves in } E \cap S_i \\ \text{mapping to } f(E)\end{pmatrix} \leq \rho(X)/\varepsilon$

$$\Longrightarrow \#\begin{pmatrix}\text{Curves } C \subseteq B_+(K_X + \varepsilon S) \cap S_i \\ \text{not contracted by } g\end{pmatrix} \leq \rho(X)^2/\varepsilon$$

Case 2: g not an iso. at the gen. pt. of C

$$\begin{tikzcd} g_j : S_i^j \arrow[r, dashed] & S_i^{j+1} \\ \overline{g_j} : \overline{S_i^j} \arrow[u] \arrow[r, dashed] & \overline{S_i^{j+1}} \arrow[u] \end{tikzcd}$$

induced on normalizations

$$N_j = \#\begin{pmatrix}\text{curves } \underline{\text{extracted}} \\ \text{by } \overline{g_j}\end{pmatrix}$$

$$\#\begin{pmatrix}\text{curves contracted} \\ \text{by } g_j\end{pmatrix} \leq \rho(\overline{S_i^j}) - \rho(S_i^{j+1}) + N_j$$

$\rightsquigarrow$ want to bound $\rho(S_i) + \sum_{j=0}^{k} N_j$

$$N_j = \#\left(\begin{array}{c}\text{flipped curves of } X^j \dashrightarrow X^{j+1} \\ \text{contained in } S_i^{j+1}\end{array}\right)$$

For Γ in $\nearrow$ let E_Γ be the exceptional of $Bl_\Gamma X^j$ that dominates Γ.

X^{j+1} terminal $\Longrightarrow$ smooth at generic point of Γ

So,

$$\begin{aligned} 0 \le a(E_\Gamma, X, \varepsilon S) &< a(E_\Gamma, X^{j+1}, \varepsilon S^{j+1}) \\ &= 1 - \varepsilon \,\mathrm{mult}_\Gamma S^{j+1} \\ &\le 1 - \varepsilon \end{aligned}$$

$$V = \{f\text{-exceptional prime divisors on } X\} \cup \left\{\begin{array}{l}\text{exceptional divisors of blow} \\ \text{ups of curves in } S_l \cap S_i,\ l \neq i\end{array}\right\}$$

$\# \mathcal{V} \leq \rho(X) + I$

Claim: $E_\Gamma \in \mathcal{V}$ for all Γ

e.g. $c_X(E_\Gamma) \neq$ pt. on X

since $a(E_\Gamma, X, \varepsilon S) = 2 - \varepsilon \, \mathrm{mult}_x S$

$\geq 2 - 3\varepsilon$

$\geq 1 - \varepsilon$ $\cancel{\times}$

So, $N_j \leq \# \mathcal{V} \leq \rho(X) + I$

$\rightsquigarrow$ only need to bound

$\#\left(X^{j+1}\text{'s that we can blow up a flipped curve of to get a valuation in } \mathcal{V} \right)$

Let $M_E^{j+1} = \mathrm{mult}_E S^{j+1}$

If E is an exceptional as above,

$$0 \leq a(E, X^{j+1}, \varepsilon S^{j+1}) = 1 - \varepsilon M_E^{j+1}$$

$$\rightsquigarrow M_E^{j+1} \leq 1/\varepsilon \quad \text{for all } j$$

$$\Longrightarrow \sum_{j=0}^{k} N_j \leq \frac{\rho(X) + I}{\varepsilon}$$

$$\rightsquigarrow \#\left(\begin{array}{l}\text{curves contained} \\ \text{in } B_+(K_X + \varepsilon S) \cap S_i\end{array}\right) \leq \frac{\rho(X)^2 + \rho(X) + I}{\varepsilon}$$

$\square$

Lemma 4.2. *Let $(X, S = \sum_{i=1}^{p} S_i)$ be a log smooth projective threefold, where $S_1, \dots, S_p$ are distinct prime divisors, and let $V = \sum_{i=1}^{p} \mathbb{R}_+ S_i \subseteq \mathrm{Div}_{\mathbb{R}}(X)$. Assume that $S_j \nsubseteq \mathbf{B}_+(K_X + B)$ for all $B \in \mathcal{L}(V)$ such that $K_X + B$ is big and for all j. Let I be the total number of irreducible components of intersections of each two of the divisors $S_1, \dots, S_p$.*

Then for any j, and for every rational number $\varepsilon > 0$ such that $(X, \varepsilon S)$ is terminal and $K_X + \varepsilon S$ is big, the number of curves contained in

$$\bigcup_{B \in \mathcal{L}_\varepsilon(V)} \mathbf{B}_+(K_X + B) \cap S_j$$

is bounded by a constant which depends on $\rho(X)$, $\rho(S_j)$, p, ε and I.

proof

Let $M = M(\varepsilon, I, \rho(X), \rho(S_j))$ bound # curves in $B_+(K_X + \varepsilon S)$.

W.L.O.G. $\varepsilon < 1/2$

$$\mathcal{L}'(V) := \{ B = \sum a_i S_i : a_i \in [\varepsilon, 1] \}$$

$B_1, \dots, B_{2^p}$ extreme points of $\mathcal{L}'(V)$

$$\rightsquigarrow \bigcup_{B \in \mathcal{L}_\varepsilon(V)} B_+(K_X + B) \subseteq \underline{\bigcup_{i=1}^{2^p} B_+(K_X + B_i)}$$

want to bound each of these

$\mathrm{mult}_{S_j}(B_i) \in \{\varepsilon, 1\} \longrightarrow 2$ cases

$\underline{\mathrm{mult}_{S_j}(B_i) = 1}$

Set $T = \varepsilon \sum_{k \neq j} S_k + S_j$

$\rightsquigarrow \left(S_j, \left(\varepsilon \sum_{k \neq j} S_k\right)\Big|_{S_j}\right)$ terminal

$f : X \dashrightarrow X'$ ample model of $K_X + T$

$S_j \notin B_+(K_X + T) = \mathrm{exc}(f) \rightsquigarrow S_j$ not contracted

$(2.6 + 2.7) \rightsquigarrow$ MMP for (X, T) restricts to an MMP for some terminal pair (S_j, Θ)

$\rightsquigarrow$ contracts $\leq \rho(S_j)$ curves

If a curve C $\underline{\text{not}}$ contracted, exists $E \supseteq C$ and that $f(E) = f(C)$

Since (X,T) is plt., $S_j' := f_* S_j$ is normal

$\Longrightarrow \operatorname{mult}_{f(C)} (S_j') = 1$

So for each f-exceptional E, there is ≤ 1 curve $C \subseteq E \cap S_i$ that maps to $f(E)$.

At most $\rho(X/X')$ such E

$\Longrightarrow$ at most $\rho(X)$ C not contracted.

$\Longrightarrow$ Curves in $B_+(K_X+T)$ bdd. by $\rho(S_j)+\rho(X)$.

$$B_+(K_X+B_i)\cap S_j \subseteq \left(B_+(K_X+T)\cup \operatorname{Supp}(B_i-T)\right)\cap S_j$$
$$\subseteq \left(B_+(K_X+T)\cup \bigcup_{k\neq j} S_k\right)\cap S_j$$

$$\#\begin{pmatrix}\text{Curves inside}\\ B_+(K_X+B_i)\cap S_j\end{pmatrix} \leq \rho(S_j)+\rho(X)+I$$

$\operatorname{mult}_{S_j}(B_i) = \varepsilon$

Since $B_i \geq \varepsilon S$

$$B_+(K_x + B_i) \cap S_j \subseteq \big(B_+(K_x + \varepsilon S) \cup B_+(B_i - \varepsilon S)\big) \cap S_j$$

$$\subseteq \Big(B_+(K_x + \varepsilon S) \cup \bigcup_{k \neq j} S_k\Big) \cap S_j$$

$$\#\begin{pmatrix} \text{curves in} \\ B_+(K_x + B_i) \cap S_j \end{pmatrix} \leq M + I$$

Lemma 4.3. *Let $(X, \sum_{i=1}^{p} S_i)$ be a 3-dimensional log smooth pair such that K_X is pseudoeffective, $S_1, \dots, S_p$ are distinct prime divisors, and let $V = \sum_{i=1}^{p} \mathbb{R}_+ S_i \subseteq \mathrm{Div}_{\mathbb{R}}(X)$. Assume that $S_i \not\subseteq \mathbf{B}(K_X + B)$ for all $B \in \mathcal{L}_\varepsilon(V)$ and every $i = 1, \dots, p$. Let $F_1, \dots, F_\ell$ be all the prime divisors contained in $\mathbf{B}(K_X)$, and for every $\nu \subseteq \{1, \dots, \ell\}$, define*

$$\mathcal{B}_\nu = \{B \in \mathcal{L}_\varepsilon^{\mathrm{can}}(V) \mid F_i \subseteq \mathbf{B}(K_X + B) \text{ if and only if } i \in \nu\}.$$

Let $\mathcal{C}_i$ be the terminal chambers in V (cf. Definition 2.15), for $1 \leqslant i \leqslant k$. Assume that each adjacent-connected component of every $\mathcal{B}_\nu$ with respect to the covering by $\mathcal{C}_i$ is the union of at most m polytopes $\mathcal{C}_i$.

Then there exists a constant $M = M(\ell, m)$ such that $k \leqslant M$.

~~proof~~

For $B \in \mathcal{L}_\varepsilon^{can}(V)$, $\mathbf{B}(K_X+B) \subseteq \mathbf{B}(K_X) \cup \mathbf{B}(B)$

$\longrightarrow$ any prime divisor in $\underline{\mathbf{B}(K_X+B)}$ is an F_j

Set $P_i = \{B \in \mathcal{L}_\varepsilon(V) : F_i \not\subseteq \mathbf{B}(K_X+B)\}$

For $\nu \subsetneq \{1, \dots, \ell\}$,

$$B_\nu = \text{closure}\Big(\bigcup_{i \notin \nu} P_i \setminus \bigcup_{j \in \nu} P_j\Big)$$

and

$$B_{\{1,\dots,\ell\}} = \text{closure}\Big(\mathcal{L}_\varepsilon^{can}(V) \setminus \bigcup_{i=1}^{\ell} P_i\Big)$$

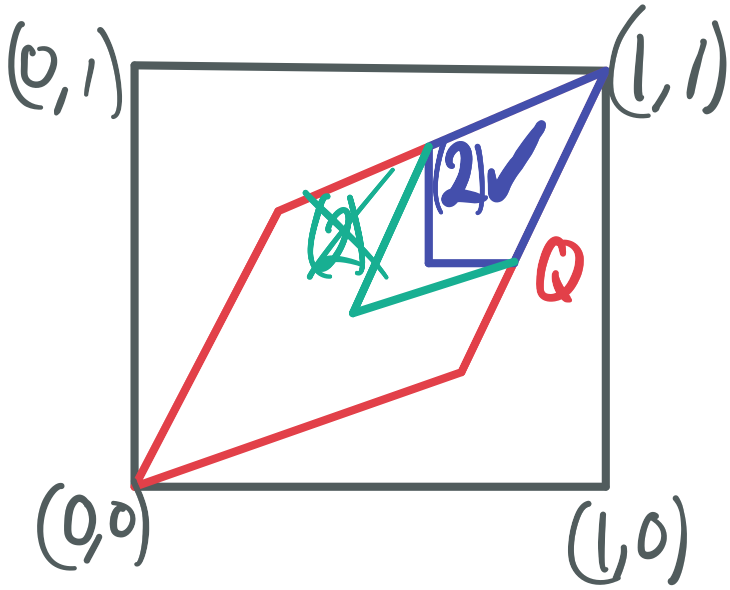
$\longrightarrow$ reduces to:

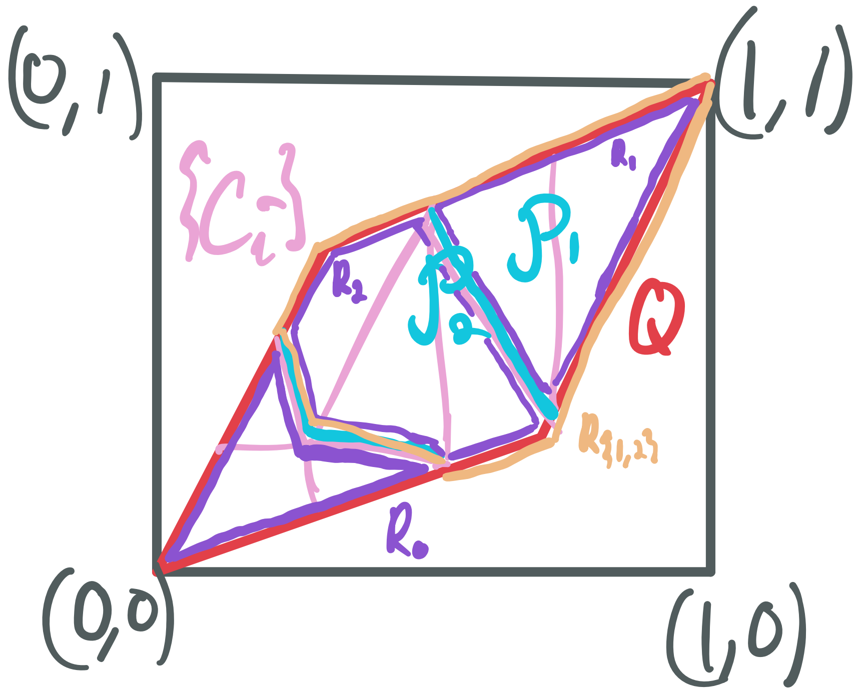
Lemma 2.11. *Let $\mathcal{Q} \subseteq [0,1]^p \subseteq \mathbb{R}^p$ be a polytope containing the origin, and let $\mathcal{C}_1, \dots, \mathcal{C}_\ell$ be p-dimensional polytopes with pairwise disjoint interiors such that $\mathcal{Q} = \bigcup_{i=1}^{\ell} \mathcal{C}_i$. Let $\mathcal{P}_1, \dots, \mathcal{P}_k \subseteq \mathcal{Q}$ be p-dimensional polytopes such that*

$$(\mathcal{P}_i + \mathbb{R}_+^p) \cap \mathcal{Q} \subseteq \mathcal{P}_i \tag{2}$$

for all i. For any subset $I \subseteq \{1, \dots, k\}$, denote by $\mathcal{R}_I$ the closure of $\bigcup_{i \in I} \mathcal{P}_i \setminus \bigcup_{j \notin I} \mathcal{P}_j$, and let $\mathcal{R}_0$ denote the closure of $\mathcal{Q} \setminus \bigcup_{i=1}^{k} \mathcal{P}_i$. Assume that each adjacent-connected component of every $\mathcal{R}_I$ and of $\mathcal{R}_0$ with respect to the covering $\mathcal{Q} = \bigcup_{i=1}^{\ell} \mathcal{C}_i$ is the union of at most m polytopes $\mathcal{C}_i$.

Then there exists a constant $M = M(k, m)$ such that $\ell \leqslant M$.





(2) $\Longrightarrow R_0$ is adjacent connected

$\rightsquigarrow$ contains $\leq m$ polytopes C_i

$$\mathcal{J}_d = \left\{ \begin{array}{c} \text{codim. } d \text{ faces of} \\ R_0 \text{ not } \subseteq \partial Q \end{array} \right\}$$

P_j contains $\leq m$ elements of $\mathcal{J}_1$

$$\Longrightarrow \#\mathcal{J}_1 \leq mk$$

Each element in $\mathcal{J}_{d-1}$ contains $\leq \#\mathcal{J}_{d-1}$ elements of $\mathcal{J}_d$ $\Longrightarrow \#\mathcal{J}_d \leq \#(\mathcal{J}_{d-1})^2$

So, $\#\mathcal{J}_d \leq (mk)^{2^{d-1}}$

$$\bigcup_{i \in I} P_i \setminus \bigcup_{j \notin I} P_j = \bigcup_{i \in I} \left(P_i \setminus \bigcup_{j \notin I} P_j \right)$$

want to bound # of C_i's in each adjacent component of

<u>Induct on k:</u> $k=1$ ✓ $\leadsto$ 2 components

By induction, assume $I = \{1, \dots, k\}$ and w.l.o.g. $i = 1$.

For $F \in \mathcal{T}_1$, set $F_1 := F \cap \mathcal{P}_1$

$$(2) \implies \hat{F}_1 := (F_1 + \mathbb{R}^p_+) \cap Q \subseteq \mathcal{P}_1$$

So, $\mathcal{P}_1 \setminus \bigcup_{j=2}^{k} \mathcal{P}_j = \bigcup_{F \in \mathcal{T}_1} \left(\hat{F}_1 \setminus \bigcup_{j=2}^{k} \mathcal{P}_j \right)$

To bound # comps. of , we can bound # comps. of $F_1 \setminus \bigcup_{j=2}^{k} \mathcal{P}_j$ with resp. to the induced topology on F_1. (2)

But, a codim-$(d-1)$ face of a comp. of $F_1 \setminus \bigcup_{j=2}^{k} \mathcal{P}_j$ is in $\mathcal{T}_d$, so

$$\# \text{comps.} \leq \text{constant}(\# \mathcal{T}_d) \qquad \square$$

$\square$

Main Technical Theorem

Theorem 4.4. *Let p and ρ be positive integers, and let ε be a positive rational number. Let $(X, \sum_{i=1}^{p} S_i)$ be a 3-dimensional log smooth pair such that*

(i) *K_X is pseudoeffective,* $\iff$ X not uniruled
(ii) *$S_1, \dots, S_p$ are distinct prime divisors which are not contained in $\mathbf{B}(K_X + B)$ for all $B \in \mathcal{L}(V)$,*
(iii) *the vector space $V = \sum_{i=1}^{p} \mathbb{R}S_i \subseteq \mathrm{Div}_{\mathbb{R}}(X)$ spans $\mathrm{Div}_{\mathbb{R}}(X)$ up to numerical equivalence,*
(iv) *$\rho(X) \leqslant \rho$ and $\rho(S_i) \leqslant \rho$ for all $i = 1, \dots, p$.*

Let I be the total number of irreducible components of intersections of each two and each three of the divisors $S_1, \dots, S_p$.

Then there exists a constant $N = N(p, \rho, \varepsilon, I)$ such that the number of terminal chambers in V which intersect the interior of $\mathcal{L}_\varepsilon(V)$ is at most N.

$\Longrightarrow$ (1.1)

Same hypotheses:

There exists a constant C that depends only on p, ρ, ε and I such that for any $\Delta = \sum_{i=1}^{p} \delta_i S_i$ with $\delta_i \in [\varepsilon, 1-\varepsilon]$ and (X, Δ) terminal, the number of log terminal models of (X, Δ) is at most C.

$\Longrightarrow$ (1.2)

Corollary 1.2. *Let ε be a positive number. Let $\mathfrak{X}$ be the collection of all log smooth 3-fold terminal pairs $(X, \Delta = \sum_{i=1}^{p} \delta_i S_i)$ such that X is not uniruled, $\varepsilon \leqslant \delta_i \leqslant 1 - \varepsilon$ for all i, $S_1, \dots, S_p$ are distinct prime divisors not contained in $\mathbf{B}(K_X + \sum_{i=1}^{p} a_i S_i)$ for all $0 \leqslant a_i \leqslant 1$, and S_i span $\mathrm{Div}_{\mathbb{R}}(X)$ up to numerical equivalence.*

Then for every $(X_0, \Delta_0) \in \mathfrak{X}$ there exists a constant N such that for every $(X, \Delta) \in \mathfrak{X}$ of the topological type as (X_0, Δ_0), the number of log terminal models of (X, Δ) is bounded by N.

~~proof~~

$\rho \leq b_2$

I is topological

Theorem 4.4. *Let p and ρ be positive integers, and let ε be a positive rational number. Let $(X, \sum_{i=1}^{p} S_i)$ be a 3-dimensional log smooth pair such that*

(i) *K_X is pseudoeffective,* $\Longleftrightarrow$ X not minimal
(ii) *$S_1, \ldots, S_p$ are distinct prime divisors which are not contained in $\mathbf{B}(K_X + B)$ for all $B \in \mathcal{L}(V)$,*
(iii) *the vector space $V = \sum_{i=1}^{p} \mathbb{R}S_i \subseteq \mathrm{Div}_{\mathbb{R}}(X)$ spans $\mathrm{Div}_{\mathbb{R}}(X)$ up to numerical equivalence,*
(iv) *$\rho(X) \leqslant \rho$ and $\rho(S_i) \leqslant \rho$ for all $i = 1, \ldots, p$.*

Let I be the total number of irreducible components of intersections of each two and each three of the divisors $S_1, \ldots, S_p$.

Then there exists a constant $N = N(p, \rho, \varepsilon, I)$ such that the number of terminal chambers in V which intersect the interior of $\mathcal{L}_\varepsilon(V)$ is at most N.

proof of 4.4

$K_X + B$ big for $B \in \mathcal{L}_\varepsilon(V)$

$$C_1, \ldots, C_q \subseteq \bigcup_{B \in \mathcal{L}_\varepsilon(V)} \mathbf{B}_+(K_X + B) \cap S$$

all curves

$$q \leq C(p, \rho, \varepsilon, I)$$

Lemma 2.3. *Let $(X, \Delta = \sum_{i=1}^{p} a_i S_i)$ be a 3-dimensional log smooth terminal pair with $0 < a_i < 1$, and let $Z \subseteq \sum_{i=1}^{p} S_i$ be a union of m curves. Let I be the total number of points of intersection of each three of the divisors $S_1, \ldots, S_p$.*

Then there exists a constant $N = N(m, p, a_1, \ldots, a_p, I)$ such that the number of geometric valuations E on X with $c_X(E) \subseteq Z$ and $a(E, X, \Delta) < 1$ is bounded by N. Furthermore, the number of blow-ups along smooth centres needed to realise the valuations is bounded by N.

sketch

Blow up to get a log smooth pair (Y, Γ) with Γ a sum of disjoint divisors

$\rightsquigarrow$ composition of $M = M(n, \rho, I)$ blow ups

$\Longrightarrow$ $\leq M$ many $E \in \mathrm{Div}(Y)$

with $a(E, X, \Delta) < 1$

To count valuations exceptional over Y, pass to a log resolution of X dominating Y and count "echos" of Y along strict transforms of curves in Z.

Alexeev, Hacon, Kawamata

"Termination of (many) 4-dim log flips"

$\square$

Finitely many geom. valuations $\{E_j\}_{j=1}^{m}$

with $c_X(E_j) \subseteq \bigcup_{i=1}^{q} C_i$ and $a(E_j, X, B) < 1$

for some $B \in \mathcal{L}_\varepsilon(V)$. $m \leq M(q, \rho, \varepsilon, I)$.

Let $F_1, \dots, F_\ell$ all prime divisors in $B(K_X)$

$\longrightarrow \ell \leq \rho$

ii) $\Longrightarrow$ For all $B \in \mathcal{L}_\varepsilon(V)$, the divisorial part of $\mathbf{B}(K_X+B)$ is contained in $\sum F_i$.

$f = f_B : X \dashrightarrow X_B$

log terminal model of (X,B)

For $\nu \subseteq \{1, \dots, \ell\}$

$\mathcal{B}_\nu = \{ B \in \mathcal{L}_\varepsilon(V) : F_i$ contracted by $f_B \}$

$\Longleftrightarrow i \in \nu$

4.3 $\rightsquigarrow$ suffice to bound # terminal chambers intersecting each adjacent-connected component of each $\mathcal{B}_\nu$.

Fix ν and assume $\mathcal{B}_\nu$ adj-connected w.l.o.g.

Set $\mu = \rho + M(q, \rho, \varepsilon, I)$

$$S = \left\{ (m_1, \dots, m_p) \in \mathbb{N}^p : m_i < \mu/\varepsilon \right\}$$

$$\mathcal{H} = \left\{ \left(\langle \Sigma_1 - \Sigma_2, \vec{x} \rangle = r \right) : \begin{array}{c} \Sigma_i \in S \\ \Sigma_1 \neq \Sigma_2 \\ r \in \mathbb{Z} \\ |r| < \mu \end{array} \right\}$$

$$\# S < 2^{\mu/\varepsilon}$$

$$\Longrightarrow \# \mathcal{H} \leq 2\mu \binom{(\mu/\varepsilon)^p}{2}$$

Elements of $\mathcal{H}$ subdivide B_v into $2^{\# \mathcal{H}}$ polytopes $\rightsquigarrow$ replace B_v with one of these

Claim: Exactly one terminal chamber has interior intersecting B_v

Suppose C', C'' are terminal chambers whose interiors intersect B_v, X', X'' their models.

Let $B \in C''_1$ and denote $(-)', (-)''$ pushforwards of divisors from X to X', X''.

For a geometric valuation E over X:

$$\Sigma_{E,C'} = (\mathrm{mult}_E S'_1, \dots, \mathrm{mult}_E S'_p)$$

$$\Sigma_{E,C''} = (\mathrm{mult}_E S''_1, \dots, \mathrm{mult}_E S''_p)$$

May assume that $X' \dashrightarrow X''$ is the flip of (X', B')

If $C \subseteq X''$ flipped, E the exceptional of $Bl_C X''$ dominating C

$$0 < a(E, X, B) < a(E, X'', B'') = 1 - \langle \Sigma_{E,C''}, b \rangle \leq 1$$

$b = (b_1, \dots, b_p)$

$B = \sum b_i S_i$

$C_X(E) \subseteq \mathrm{Supp}(\sum S_i)$

since $0 \underset{\substack{(X,B)\\ \text{terminal}}}{<} a(E,X,B) < 1$

$X'' = \mathrm{Proj}\, R(X, K_X+B)$ since $B \in \mathrm{int}(C'')$

so $C_X(E) \subseteq B_+(K_X+B) = \mathrm{exc}(X \dashrightarrow X'')$

$\Longrightarrow E$ is one of the $E_1, \dots, E_m$ above

Also,

$0 < a(E, X', B') = \mu_{E,B} - \langle \Sigma_{E,C'}, b \rangle$

for some $0 < \mu_{E,B} < \mu$, $\mu_{E,B} \in \mathbb{Z}$

$b_i \geq \varepsilon \Longrightarrow 0 \leq \mathrm{mult}_E S_i' < \mu/\varepsilon$

So $\Sigma_{E,C'} \in S$.

If $B \in C' \cap C''$,

$$\text{Negativity} \Longrightarrow a(E, X'; B') = a(E, X''; B'')$$

$$\mu_{E,B} - \langle \Sigma_{E,C'}, b \rangle = a(E, X'; B') = a(E, X''; B'')$$

$$= 1 - \langle \Sigma_{E,C''}, b \rangle$$

$$\left(\langle \Sigma_{E,C'} - \Sigma_{E,C''}, b \rangle = \mu_{E,B} - 1 \right) \in \mathcal{H}$$

$$\Longrightarrow \Sigma_{E,C'} = \Sigma_{E,C''}$$

If $B \in \operatorname{int}(C'')$

$$\text{Negativity} \Longrightarrow a(E, X'; B') < a(E, X''; B'')$$

$$\mu_{E,B} - \langle \Sigma_{E,C'}, b \rangle = a(E, X'; B') < a(E, X''; B'')$$

$$= 1 - \langle \Sigma_{E,C''}, b \rangle \leq 1$$

$\mu_{E,B} < 1$

$\square$